

## RESEARCH PAPER

### MULTI-TERM FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN POWER GROWTH FUNCTION SPACES

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#### Abstract

In this paper we characterize the Laplace transform of functions with power growth square averages and study several multi-term Caputo and Riemann-Liouville fractional integro-differential equations in this space of functions.

*MSC 2010:* Primary: 44A10, 26A33; Secondary: 45J05

*Key Words and Phrases:* functions with square averages power growth; Laplace transform; Caputo fractional derivative; Riemann-Liouville fractional derivative; fractional integro-differential equation

#### 1. Introduction

The use of fractional differential and integral operators in mathematical models has become increasingly widespread in recent years, mainly due to the fact that fractional order differential and integral operators are nonlocal in nature and help to gain insight into the hereditary of the associated phenomena. Examples include long-time memory in financial time series via fractional Langevin equations [18], chaotic neuron model [16], transport of passive tracers carried by fluid flow in a porous medium in groundwater hydrology [19], viscoelastic behavior [10], etc. Further examples can be found, for example, in [9, 13].

In [22] fractional integro-differential equations have been considered in a class of power growth functions. Denote [22]  $f \in BSA(\mathbb{R}_+)$  if  $f$  is locally integrable on  $\mathbb{R}_+$  and

$$\sup_{T>0} \frac{1}{T+1} \int_0^T |f(t)|^2 dt < \infty.$$

For the following Caputo and Riemann-Liouville fractional integro-differential equations

$${}^C\partial_t^\alpha f(t) + kf(t) + \int_0^t g(t-\tau)f(\tau)d\tau = h(t), \quad f(0) = f_0, \quad (1.1)$$

$$D_{0+}^\alpha f(t) + kf(t) + \int_0^t g(t-\tau)f(\tau)d\tau = h(t), \quad I_{0+}^{1-\alpha} f(0+) = f_0, \quad (1.2)$$

where  ${}^C\partial_t^\alpha$ ,  $D_{0+}^\alpha$ , and  $I_{0+}^{1-\alpha}$  are the Caputo and Riemann-Liouville fractional derivatives and the Riemann-Liouville fractional integral [7], it was shown [22] that if  $g, h \in L^1(\mathbb{R}_+)$ ,  $\frac{1}{2} < \alpha \leq 1$ , and  $\|g\|_1 < k$ , then the Caputo fractional integro-differential equation (1.1) and the Riemann-Liouville fractional integro-differential equation (1.2) have unique solutions  $f$  from  $BSA(\mathbb{R}_+)$ .

The aim of this paper is to extend the results obtained in [22] for single-term equations (1.1) and (1.2) to wider classes of multi-term fractional integro-differential equations

$$D_{0+}^{\alpha_0} f(t) + \sum_{j=1}^n a_j D_{0+}^{\alpha_j} f(t) + kf(t) + \int_0^t g(t-\tau)f(\tau)d\tau = h(t), \\ I_{0+}^{1-\alpha_0} f(0+) = f_0, \quad (1.3)$$

$${}^C\partial_t^{\alpha_0} f(t) + \sum_{j=1}^n a_j {}^C\partial_t^{\alpha_j} f(t) + kf(t) + \int_0^t g(t-\tau)f(\tau)d\tau = h(t), \\ f(0+) = f_0, \quad (1.4)$$

$${}^C\partial_t^{\alpha_0} f(t) + \sum_{j=1}^n a_j {}^C\partial_t^{\alpha_j} f(t) + \sum_{j=1}^m b_j D_{0+}^{\beta_j} f(t) + kf(t) \\ + \int_0^t g(t-\tau)f(\tau)d\tau = h(t), \quad f(0+) = f_0, \quad (1.5)$$

$$\frac{1}{2} < \alpha_0 \leq 1, \quad 0 < \alpha_n < \dots < \alpha_1 < \alpha_0, \quad 0 < \beta_m < \dots < \beta_1 < \alpha_0,$$

where  $g, h \in L^1(\mathbb{R}_+)$ ,  $a_1, \dots, a_n, b_1, \dots, b_m, k \in \mathbb{R}_+$ , are given, and  $f$  is an unknown. It turns out that solutions of equations (1.3), (1.4) and (1.5) have different power growth, and in fact (1.3) has a solution in a subspace of  $BSA(\mathbb{R}_+)$ , while equations (1.4) and (1.5) have no solution in  $BSA(\mathbb{R}_+)$ .

Multi-term fractional differential equations (equations (1.3), (1.4) with  $g = 0$ ) play an important role in modeling of motion of a rigid plate immersed in a Newtonian fluid. These problems arise, for instance, in the Basset equation [14] and the Bagley Torvik equation [20]. One can find some recent works on multi-term fractional differential equations in [2, 3, 4, 8, 11, 12].

## 2. Functions with square average power growth

Denote by  $\mathcal{L}$  and  $\mathcal{L}^{-1}$  the Laplace transform and its inverse transform, [23],

$$\begin{aligned} F(s) &= (\mathcal{L}f)(s) := \int_0^\infty e^{-st} f(t) dt, \\ f(t) &= (\mathcal{L}^{-1}F)(t) := \frac{1}{2\pi i} \int_{\text{Res}=d} F(s) e^{st} ds. \end{aligned} \quad (2.1)$$

It is well known [23] that if  $f(t)$  is locally integrable and has a power growth, then  $F(s)$  exists and is holomorphic in the right-half plane  $\text{Re } s > 0$ . The Tauberian theorem for the Laplace transform [23]

$$f(t) \sim \frac{At^{p-1}}{\Gamma(p)}, \quad t \rightarrow \infty \implies F(s) \sim \frac{A}{s^p} \quad s \rightarrow 0_+ \quad p > 0, \quad (2.2)$$

says that, moreover, if  $f(t)$  grows as  $t^{p-1}$  at infinity, then  $F(s)$  grows as  $s^{-p}$  at 0.

The converse question is if  $F(s)$  is holomorphic in the right-half plane  $\text{Re } s > 0$ , and has a power growth at 0, whether it is the Laplace transform of a power growth function. The answer turns out affirmative if we consider functions of square average power growth instead of functions with pointwise power growth.

We now generalize the Wiener class of functions  $BSA(\mathbb{R}_+)$ , introduced in [21, 22], to functions with square average power growth on  $\mathbb{R}_+ = (0; \infty)$ .

**DEFINITION 2.1.** By  $BSA_p(\mathbb{R}_+)$ , the linear space of functions with square average power growth of order  $p \geq 0$ , we denote the set of locally integrable functions  $f$  on  $\mathbb{R}_+$  such that

$$\sup_{T>0} \frac{1}{(T+1)^p} \int_0^T |f(t)|^2 dt < \infty, \quad (2.3)$$

and

$$BSA_\infty(\mathbb{R}_+) = \bigcup_{p>0} BSA_p(\mathbb{R}_+). \quad (2.4)$$

We say  $f \in BSA_p^m(\mathbb{R}_+)$  if  $f, f', \dots, f^{(m)} \in BSA_p(\mathbb{R}_+)$ .

Clearly,  $BSA_0(\mathbb{R}_+) = L^2(\mathbb{R}_+)$ ,  $BSA_1(\mathbb{R}_+) = BSA(\mathbb{R}_+)$ , and  $BSA_p(\mathbb{R}_+) \subset BSA_{p'}(\mathbb{R}_+)$  if  $p < p'$ . It is readily seen that  $L^2(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+) \subset BSA_p(\mathbb{R}_+)$ ,  $p \geq 1$ , and by Hölder's inequality  $L^q(\mathbb{R}_+) \subset BSA_p(\mathbb{R}_+)$ , for  $2 \leq q \leq \infty$ ,  $p \geq 1$ . However, note that, for  $p \geq 0$ , we have  $f(t) = t^p \in BSA_{2p+1}(\mathbb{R}_+)$ , and yet  $f(t) \notin L^q(\mathbb{R}_+)$ ,  $0 < q < \infty$ .

Now we characterize the Laplace transform of functions from  $BSA_p(\mathbb{R}_+)$ .

**THEOREM 2.1.** *A function  $F(s)$  is the Laplace transform of  $f \in BSA_p(\mathbb{R}_+)$  if, and only if,  $F(s)$  is holomorphic in the right-half plane  $\operatorname{Re} s > 0$ , and*

$$\sup_{x>0} \left( \frac{x}{x+1} \right)^p \int_{-\infty}^{\infty} |F(x+iy)|^2 dy < \infty. \quad (2.5)$$

**P r o o f.** The case  $p = 0$  is the Paley-Wiener theorem for the Laplace transform [17, 23]

$$f(t) \in L^2(\mathbb{R}_+) \iff F(s) \text{ is holomorphic in } \operatorname{Re} s > 0, \\ \sup_{x>0} \int_{-\infty}^{\infty} |F(x+iy)|^2 dy < \infty. \quad (2.6)$$

For  $p > 0$  the proof is similar to the proof of Theorem 2.3 in [22], and therefore, detailed steps are omitted. Let  $f \in BSA_p(\mathbb{R}_+)$ . Denote  $\tilde{f}(T) = \int_0^T f(t) dt$ , then by the Hölder inequality  $|\tilde{f}(T)| \leq C\sqrt{T}(T+1)^{\frac{p}{2}}$  and integration by parts yields

$$F(s) = s \int_0^{\infty} e^{-st} \tilde{f}(t) dt, \quad \operatorname{Re} s > 0.$$

Thus  $F(s)$  is holomorphic in the right half plane  $\operatorname{Re} s > 0$ .

Integration by parts gives

$$\begin{aligned} \int_0^{\infty} e^{-2xt} |f(t)|^2 dt &= 2x \int_0^{\infty} e^{-2xT} \int_0^T |f(t)|^2 dt dT \\ &\leq Cx \int_0^{\infty} (T+1)^p e^{-2xT} dT = \frac{Ce^{2x}}{2^{p+1}x^p} \int_{2x}^{\infty} \tau^p e^{-\tau} d\tau \\ &= \frac{Ce^{2x}}{2^{p+1}x^p} \Gamma(p+1, 2x), \end{aligned} \quad (2.7)$$

where  $\Gamma(p+1; 2x)$  is the upper incomplete Gamma function [1]. Using the asymptotics of the upper incomplete Gamma function [1]

$$\begin{aligned}\Gamma(p, x) &\sim x^{p-1}e^{-x}, & x \rightarrow \infty, \\ \Gamma(p, x) &\sim \Gamma(p), & x \rightarrow 0,\end{aligned}$$

we see that the last expression of (2.7) is bounded at infinity and  $\sim x^{-p}$  at 0. Consequently,

$$\int_0^\infty e^{-2xt}|f(t)|^2 dt \leq C\left(\frac{x+1}{x}\right)^p. \quad (2.8)$$

Hence,  $e^{-xt}f(t) \in L^2(\mathbb{R}_+)$  for any  $x > 0$ . Consequently,  $F(s)$  with  $\operatorname{Re} s > x > 0$  is the Laplace transform of  $e^{-xt}f(t) \in L^2(\mathbb{R}_+)$  at the point  $s - x$ . The Parseval formula for the Laplace transform in  $L^2(\mathbb{R}_+)$ , see [23], gives

$$\int_{-\infty}^\infty |F(x+iy)|^2 dy = 2\pi \int_0^\infty e^{-2xt}|f(t)|^2 dt \leq \frac{C(x+1)^p}{x^p}, \quad x > 0,$$

that yields (2.5).

Conversely, assume that  $F(s)$  is holomorphic in the right-half plane  $\operatorname{Re} s > 0$  and formula (2.5) holds. Similar to the proof of [22, Theorem 2.3], one concludes that  $F$  is the Laplace transform of some  $f \in L^2(\mathbb{R}_+)$ , and the Parseval formula for the Laplace transform yields

$$\int_0^\infty e^{-2xt}|f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty |F(x+iy)|^2 dy \leq \frac{C(x+1)^p}{x^p}, \quad x > 0.$$

Take

$$g(t) = \begin{cases} \frac{1}{t}, & t > e^{-2} \\ 0, & 0 < t \leq e^{-2} \end{cases}.$$

Then  $\|g\|_\infty = e^2$ , and

$$\begin{aligned}\int_0^{1/x} |f(t)|^2 dt &= \int_0^\infty e^{-2xt} g(e^{-2xt}) |f(t)|^2 dt \\ &\leq \|g\|_\infty \int_0^\infty e^{-2xt} |f(t)|^2 dt \leq \frac{C(x+1)^p}{x^p}, \quad x > 0.\end{aligned}$$

Replacing  $x$  by  $\frac{1}{T}$  we arrive at (2.3), and Theorem 2.1 is proved.  $\square$

### 3. Special cases

**COROLLARY 3.1.** *Let  $F(s)$  be holomorphic in the right half plane  $\operatorname{Re} s > 0$  and  $|F(s)| \leq C|s|^{-\alpha}$ ,  $\alpha > \frac{1}{2}$ . Then  $F$  is the Laplace transform of a function  $f \in BSA_{2\alpha-1}(\mathbb{R}_+)$ .*

P r o o f. Because  $\alpha > \frac{1}{2}$ ,  $F(x + i\bullet) \in L^2(\mathbb{R})$ , and

$$\begin{aligned} \frac{x^{2\alpha-1}}{(x+1)^{2\alpha-1}} \int_{-\infty}^{\infty} |F(x+iy)|^2 dy &\leq \frac{Cx^{2\alpha-1}}{(x+1)^{2\alpha-1}} \int_{-\infty}^{\infty} (x^2+y^2)^{-\alpha} dy \\ &= \frac{C\sqrt{\pi}\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)(x+1)^{2\alpha-1}} < \infty, \end{aligned}$$

hence, formula (2.5) holds, i.e.,  $f \in BSA_{2\alpha-1}(\mathbb{R}_+)$ .  $\square$

The following result explains the importance of  $BSA_p(\mathbb{R}_+)$  in studying fractional calculus.

THEOREM 3.1. *Let  $g \in L^1(\mathbb{R}_+)$  with  $\|g\|_1 < k$ ,  $0 < \alpha \leq 1$ . Then*

$$\frac{1}{|s^\alpha + k + G(s)|} \leq \frac{4k}{k - \|g\|_1} |s|^{-\alpha}, \quad \operatorname{Re} s > 0. \quad (3.1)$$

If, moreover,  $\frac{1}{2} < \alpha \leq 1$ , then the inverse Laplace transform

$$f := \mathcal{L}^{-1} \left( \frac{1}{s^\alpha + k + G(s)} \right) \quad (3.2)$$

is from  $BSA_{2\alpha-1}(\mathbb{R}_+)$ .

P r o o f. Since  $g \in L^1(\mathbb{R}_+)$ , its Laplace transform  $G(s)$  is holomorphic in the right half-plane, and from

$$|G(s)| \leq \int_0^\infty e^{-(\operatorname{Re} s)t} |g(t)| dt \leq \|g\|_1 < k, \quad \operatorname{Re}(s^\alpha) > 0, \quad \text{for } \operatorname{Re} s > 0,$$

we deduce that  $\operatorname{Re}(s^\alpha + k + G(s)) > 0$  when  $\operatorname{Re} s > 0$ . Consequently,  $\frac{1}{s^\alpha + k + G(s)}$  is also holomorphic in the right half-plane. Let us denote by

$$h(s) = k + G(s),$$

then  $h(s)$  is clearly holomorphic in the right half-plane, and for  $\operatorname{Re} s > 0$ ,

$$0 < k - \|g\|_1 \leq \operatorname{Re} h(s) \quad (3.3)$$

and also  $|h(s)| < 2k$ .

For  $|s|^\alpha > 4k$ ,  $\operatorname{Re} s > 0$ , we have

$$|s^\alpha + h(s)| \geq |s|^\alpha - |h(s)| > |s|^\alpha - 2k > \frac{1}{2}|s|^\alpha > \frac{k - \|g\|_1}{4k} |s|^\alpha. \quad (3.4)$$

For  $|s|^\alpha \leq 4k$ ,  $\operatorname{Re} s > 0$ , we have

$$|s^\alpha + h(s)| \geq \operatorname{Re}(s^\alpha + h(s)) > k - \|g\|_1 > \frac{k - \|g\|_1}{4k} |s|^\alpha. \quad (3.5)$$

Combining (3.4) and (3.5) we obtain (3.1). Statement (3.2) follows from Corollary 3.1.  $\square$

REMARK 3.1. If  $\frac{1}{2} < \alpha \leq 1$ , then  $BSA_{2\alpha-1}(\mathbb{R}_+) \subset BSA_1(\mathbb{R}_+) = BSA(\mathbb{R}_+)$ , hence, Corollary 3.1 and Theorem 3.1 improve [22, Corollary 2.1] and [22, Theorem 2.2], respectively.

Combining Corollary 3.1 and Theorem 3.1 we arrive at

COROLLARY 3.2. Let  $\|g\|_1 < k, 0 < \alpha \leq 1, \beta < \alpha - \frac{1}{2}$ , then

$$\mathcal{L}^{-1} \left( \frac{s^\beta}{s^\alpha + k + G(s)} \right) \in BSA_{2(\alpha-\beta)-1}(\mathbb{R}_+). \quad (3.6)$$

As an example, consider the two-parametric Mittag-Leffler function ([6])

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha > 0.$$

We have ([6])

$$\mathcal{L}(t^{\beta-1} E_{\alpha,\beta}(-kt^\alpha))(s) = \frac{s^{\alpha-\beta}}{s^\alpha + k}.$$

Consequently, by Corollary 3.2 if  $k > 0, 0 < \alpha \leq 1$ , and  $\beta > \frac{1}{2}$ , then  $t^{\beta-1} E_{\alpha,\beta}(-kt^\alpha) \in BSA_{2\beta-1}(\mathbb{R}_+)$ .

The following theorem plays a crucial role in studying multi-term fractional integro-differential equations in the next sections.

THEOREM 3.2. Let  $g \in L^1(\mathbb{R}_+)$  with  $\|g\|_1 < k, 0 < \alpha_n < \dots < \alpha_1 < \alpha_0 \leq 1$ , and  $a_1, a_2, \dots, a_n > 0$ . Then

$$\frac{1}{\left| s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s) \right|} \leq \frac{C_1^{\alpha_0}}{k - \|g\|_1} |s|^{-\alpha_0}, \quad \operatorname{Re} s > 0, \quad (3.7)$$

where

$$C_1 = \max_{1 \leq j \leq n} \left\{ [(n+2)a_j]^{\frac{1}{\alpha_0 - \alpha_j}}, [2k(n+2)]^{\frac{1}{\alpha_0}} \right\}. \quad (3.8)$$

If, moreover,  $\frac{1}{2} < \alpha_0 \leq 1$ , then the inverse Laplace transform

$$\mathcal{L}^{-1} \left( \frac{1}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)} \right) \in BSA_{2\alpha_0-1}(\mathbb{R}_+). \quad (3.9)$$

**P r o o f.** As in the proof of Theorem 3.1,  $G(s)$  is holomorphic in the right half-plane  $\operatorname{Re} s > 0$ , and there  $\operatorname{Re}(k + G(s)) \geq k - \|g\|_1 > 0$ . Since  $0 < \alpha_j \leq 1$ , then  $\operatorname{Re}(s^{\alpha_j}) > 0, j = 0, 1, 2, \dots, n$ , in  $\operatorname{Re} s > 0$ . Together with  $a_1, a_2, \dots, a_n > 0$  we arrive at

$$\operatorname{Re} \left( s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s) \right) > 0,$$

in  $\operatorname{Re} s > 0$ . Consequently,  $\frac{1}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}$  is also holomorphic in the right half-plane. For  $|s| > C_1, \operatorname{Re} s > 0$ , we have

$$\frac{|s|^{\alpha_0}}{n+2} - a_j |s|^{\alpha_j} \geq 0, \quad j = 1, 2, 3, \dots, n, \quad \frac{|s|^{\alpha_0}}{n+2} - k - |G(s)| \geq 0.$$

Consequently,

$$\begin{aligned} \left| s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s) \right| &\geq |s|^{\alpha_0} - \sum_{j=1}^n a_j |s|^{\alpha_j} - k - |G(s)| \quad (3.10) \\ &\geq \frac{|s|^{\alpha_0}}{n+2} + \sum_{j=1}^n \left( \frac{|s|^{\alpha_0}}{n+2} - a_j |s|^{\alpha_j} \right) + \left( \frac{|s|^{\alpha_0}}{n+2} - k - |G(s)| \right) > \frac{|s|^{\alpha_0}}{n+2}. \end{aligned}$$

For  $|s| \leq C_1, \operatorname{Re} s > 0$ ,

$$\begin{aligned} \left| s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s) \right| &\geq \operatorname{Re} \left( s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s) \right) \quad (3.11) \\ &> \operatorname{Re}(k + G(s)) \geq k - \|g\|_1 \geq \frac{k - \|g\|_1}{C_1^{\alpha_0}} |s|^{\alpha_0}. \end{aligned}$$

Since  $C_1^{\alpha_0} \geq 2k(n+2)$  we have

$$\frac{k - \|g\|_1}{C_1^{\alpha_0}} \leq \frac{k}{2k(n+2)} < \frac{1}{n+2}.$$

Combining (3.10) and (3.11) we obtain (3.7). Statement (3.9) follows from Corollary 3.1.  $\square$

**LEMMA 3.1.** *Let  $f \in BSA_p(\mathbb{R}_+)$  and  $g \in L^1(\mathbb{R}_+)$ . Then their Laplace convolution*

$$h(t) = (f * g)(t) := \int_0^t f(t - \tau) g(\tau) d\tau \quad (3.12)$$

*belongs to  $BSA_p(\mathbb{R}_+)$ .*

**P r o o f.** In fact, applying the Laplace transform to (3.12), we obtain  $H(s) = F(s)G(s)$ , therefore,  $|H(s)| \leq |F(s)| \|g\|_1$ , and

$$\begin{aligned} & \sup_{x>0} \left( \frac{x}{x+1} \right)^p \int_{-\infty}^{\infty} |H(x+iy)|^2 dy \\ & \leq \|g\|_1^2 \sup_{x>0} \left( \frac{x}{x+1} \right)^p \int_{-\infty}^{\infty} |F(x+iy)|^2 dy < \infty. \end{aligned} \quad (3.13)$$

□

#### 4. Multi-term Riemann-Liouville fractional integro-differential equation

Consider now the following multi-term Riemann-Liouville fractional integro-differential equation

$$D_{0+}^{\alpha_0} f(t) + \sum_{j=1}^n a_j D_{0+}^{\alpha_j} f(t) + k f(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \quad (4.1)$$

$$I_{0+}^{1-\alpha_0} f(0+) = f_0, \quad \frac{1}{2} < \alpha_0 \leq 1, \quad 0 < \alpha_n < \dots < \alpha_1 < \alpha_0,$$

where  $k, a_1, a_2, \dots, a_n \in \mathbb{R}_+$ ,  $g, h \in L^1(\mathbb{R}_+)$  are given, and  $f$  is an unknown. Here  $D_{0+}^{\alpha}$  is the Riemann-Liouville fractional derivative ([7])

$$D_{0+}^{\alpha} f(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} f(t), \quad I_{0+}^{n-\alpha} f(t) = \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(\tau) d\tau, \quad \alpha < n. \quad (4.2)$$

Special cases of (4.1) have been considered in [7].

It is well known ([7]) that

$$\mathcal{L}(D_{0+}^{\alpha} f)(s) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} D_{0+}^{\alpha+k-n} f(0+), \quad n-1 \leq \alpha < n. \quad (4.3)$$

**THEOREM 4.1.** *Let  $k > 0$ ,  $f_0 \in \mathbb{R}$ ,  $g, h \in L^1(\mathbb{R}_+)$ , be given, and  $\|g\|_1 < k$ . Then the multi-term Riemann-Liouville fractional integro-differential equation (4.1) has a unique solution  $f$  from  $BSA_{2\alpha_0-1}(\mathbb{R}_+)$ .*

**P r o o f.** Since  $I_{0+}^{1-\alpha_0} f(0+) = f_0$ , and  $1-\alpha_0 < 1-\alpha_j$ , then  $I_{0+}^{1-\alpha_j} f(0+) = 0$ ,  $j = 1, 2, \dots, n$ . Applying the Laplace transform to equation (4.1) and taking into account (4.3) we obtain

$$s^{\alpha_0} F(s) - f_0 + \sum_{j=1}^n a_j s^{\alpha_j} F(s) + k F(s) + G(s) F(s) = H(s). \quad (4.4)$$

Solving for  $F(s)$  yields

$$F(s) = \frac{f_0 + H(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}. \quad (4.5)$$

Denote

$$M(s) = \frac{1}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}, \quad (4.6)$$

then according to Theorem 3.2, its inverse Laplace transform, namely  $m(t)$ , belongs to  $BSA_{2\alpha_0-1}(\mathbb{R}_+)$ , and

$$f(t) = f_0 m(t) + \int_0^t m(t-\tau) h(\tau) d\tau. \quad (4.7)$$

Since  $m \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$  and  $h \in L^1(\mathbb{R}_+)$ , by Lemma 3.1, their Laplace convolution  $m * h$  belongs to  $BSA_{2\alpha_0-1}(\mathbb{R}_+)$ . Hence,  $f$ , defined by (4.7), is from  $BSA_{2\alpha_0-1}(\mathbb{R}_+)$ . Using the Tauberian theorem for the Laplace transform ([23]) we have

$$M(s) \sim \frac{1}{s^{\alpha_0}}, \quad s \rightarrow \infty \implies m(t) \sim \frac{t^{\alpha_0-1}}{\Gamma(\alpha_0)}, \quad t \rightarrow 0+. \quad (4.8)$$

Consequently,  $I_{0+}^{1-\alpha_0} m(t) \sim 1$ ,  $t \rightarrow 0+$ . Together with (4.7) it yields  $I_{0+}^{1-\alpha_0} f(0+) = f_0$ .

Conversely, let  $f$  be given by (4.7), where  $m$  is defined as the Laplace inverse of (4.6). Then  $f \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$  and  $I_{0+}^{1-\alpha_0} f(0+) = f_0$ . Applying the Laplace transform to (4.7) and taking into account (4.6) we arrive at (4.5). Hence, (4.4) holds. The Laplace inverse of (4.4) yields (4.1).  $\square$

**REMARK 4.1.** Theorem 4.1 from [22] proves that a special case of equation (4.1) (when  $a_1 = a_2 = \dots = a_n = 0$ ) under the same conditions of Theorem 4.1 has a unique solution in  $BSA(\mathbb{R}_+) = BSA_1(\mathbb{R}_+) \supset BSA_{2\alpha_0-1}(\mathbb{R}_+)$ . Thus, Theorem 4.1 generalizes and improves [22, Theorem 4.1].

## 5. Multi-term Caputo fractional integro-differential equation

Consider the following Caputo fractional integro-differential equation

$${}^C \partial_t^{\alpha_0} f(t) + \sum_{j=1}^n a_j {}^C \partial_t^{\alpha_j} f(t) + k f(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \quad f(0+) = f_0, \quad (5.1)$$

$$\frac{1}{2} < \alpha_0 \leq 1, \quad 0 < \alpha_n < \dots < \alpha_1 < \alpha_0,$$

where  $a_j, k \in \mathbb{R}_+, j = 1, 2, \dots, n$ , and  $g, h \in L^1(\mathbb{R}_+)$  are given, and  $f$  is an unknown. Here  ${}^C\partial_t^\alpha$  is the Caputo fractional derivative ([7])

$${}^C\partial_t^\alpha f(t) = \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(\tau) d\tau, \quad n-1 < \alpha < n, \quad {}^C\partial_t^n f(t) = f^{(n)}(t). \quad (5.2)$$

It is well known [7] that

$$\mathcal{L}({}^C\partial_t^\alpha f)(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha \leq n. \quad (5.3)$$

**THEOREM 5.1.** *Let  $k > 0, f_0 \in \mathbb{R}, g, h \in L^1(\mathbb{R}_+)$ , be given, and  $\|g\|_1 < k$ . Then the Caputo fractional integro-differential equation (5.1) has a unique solution  $f$  from  $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$ .*

**P r o o f.** Applying the Laplace transform to equation (5.1) and taking into account (5.3) we obtain

$$(s^{\alpha_0} F(s) - s^{\alpha_0-1} f_0) + \sum_{j=1}^n a_j (s^{\alpha_j} F(s) - s^{\alpha_j-1} f_0) + k F(s) + G(s) F(s) = H(s). \quad (5.4)$$

Solving for  $F(s)$  yields

$$F(s) = \frac{s^{\alpha_0-1} f_0 + f_0 \sum_{j=1}^n a_j s^{\alpha_j-1} + H(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}. \quad (5.5)$$

Denote

$$\begin{aligned} L_j(s) &= \frac{s^{\alpha_j-1}}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}, \quad j = 0, 1, \dots, n, \\ M(s) &= \frac{1}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}, \end{aligned} \quad (5.6)$$

then according to Theorem 3.2, their inverse Laplace transforms, namely  $l_j(t)$  and  $m(t)$ , belong to  $BSA_{2(\alpha_0-\alpha_j)+1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$ , and  $BSA_{2\alpha_0-1}(\mathbb{R}_+)$ , respectively. Moreover,

$$f(t) = f_0 l_0(t) + f_0 \sum_{j=1}^n a_j l_j(t) + \int_0^t m(t-\tau) h(\tau) d\tau. \quad (5.7)$$

Since  $m \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$  and  $h \in L^1(\mathbb{R}_+)$ , by Lemma 3.1, their Laplace convolution  $m*h$  belongs to  $BSA_{2\alpha_0-1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$ . Hence,  $f$ , defined by (5.7), is from  $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$ . Using the Tauberian theorem for the Laplace transform [23] we have

$$L_0(s) \sim \frac{1}{s}, \quad s \rightarrow \infty \implies l_0(t) \sim 1, \quad t \rightarrow 0+,$$

and

$$\begin{aligned} L_j(s) &\sim \frac{1}{s^{\alpha_0 - \alpha_j + 1}}, \quad s \rightarrow \infty, \\ \implies l_j(t) &\sim \frac{t^{\alpha_0 - \alpha_j}}{\Gamma(\alpha_0 - \alpha_j + 1)}, \quad t \rightarrow 0+, \quad j = 1, \dots, n. \end{aligned}$$

Consequently,  $f(0+) = f_0$ .

Conversely, let  $f$  be given by (5.7), where  $l_j, m$  are defined as the Laplace inverse transforms of (5.6). Then  $f \in BSA_{2(\alpha_0 - \alpha_n) + 1}(\mathbb{R}_+)$  and  $f(0+) = f_0$ . Applying the Laplace transform to (5.7) and taking into account (5.6) we arrive at (5.5). Hence, (5.4) holds. The Laplace inverse transform of (5.4) yields (5.1).  $\square$

REMARK 5.1. If  $f_0 = 0$ , then the Riemann-Liouville derivative and the Caputo derivative coincide, and the solution  $f$  of (5.1) also belongs to  $BSA_{2\alpha_0 - 1}(\mathbb{R}_+)$ .

REMARK 5.2. If  $n = 0$ , then  $f \in BSA_1(\mathbb{R}_+)$ . Therefore, Theorem 5.1 generalizes [22, Theorem 3.1].

REMARK 5.3. Although  $g, h \in L^1(\mathbb{R}_+)$ , in general  $f \notin L^1(\mathbb{R}_+)$ . In fact, if  $f \in L^1(\mathbb{R}_+)$ , then  $|F(s)| \leq \|f\|_1$  for  $\operatorname{Re} s \geq 0$ . But if  $f_0 \neq 0$ , then from (5.5) we have

$$F(s) \sim Cs^{\alpha_n - 1} \rightarrow \infty,$$

as  $s \rightarrow 0+$ . Consequently,  $f \notin L^1(\mathbb{R}_+)$ .

## 6. Mixed Caputo Riemann-Liouville fractional integro-differential equation

Now we consider the following mixed Caputo Riemann-Liouville fractional integro-differential equation with a dominant Caputo fractional derivative

$$\begin{aligned} {}^C\partial_t^{\alpha_0} f(t) + \sum_{j=1}^n a_j {}^C\partial_t^{\alpha_j} f(t) + \sum_{j=1}^m b_j D_{0+}^{\beta_j} f(t) + kf(t) \\ + \int_0^t g(t-\tau)f(\tau)d\tau = h(t), \quad f(0+) = f_0, \\ \frac{1}{2} < \alpha_0 \leq 1, \quad 0 < \alpha_n < \dots < \alpha_1 < \alpha_0, \quad 0 < \beta_m < \dots < \beta_1 < \alpha_0, \end{aligned} \quad (6.1)$$

where  $g, h \in L^1(\mathbb{R}_+)$ ,  $a_1, \dots, a_n, b_1, \dots, b_m, k \in \mathbb{R}_+$ , are given, and  $f$  is an unknown.

**THEOREM 6.1.** *Let  $k > 0$ ,  $f_0 \in \mathbb{R}$ ,  $g, h \in L^1(\mathbb{R}_+)$ , be given, and  $\|g\|_1 < k$ . Then the mixed Caputo Riemann-Liouville fractional integro-differential equation (6.1) has a unique solution  $f$  from  $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$ .*

**P r o o f.** Since  $f(0+) = f_0$ , then  $I_{0+}^{1-\beta_j}(0+) = 0$ ,  $j = 1, \dots, m$ , and applying the Laplace transform to equation (6.1) and taking into account (4.1) and (5.3), we obtain

$$(s^{\alpha_0} F(s) - s^{\alpha_0-1} f_0) + \sum_{j=1}^n a_j (s^{\alpha_j} F(s) - s^{\alpha_j-1} f_0) + \sum_{j=1}^m b_j s^{\beta_j} F(s) + kF(s) + G(s)F(s) = H(s). \quad (6.2)$$

Solving for  $F(s)$  yields

$$F(s) = \frac{f_0 s^{\alpha_0-1} + f_0 \sum_{j=1}^n a_j s^{\alpha_j-1} + H(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + \sum_{j=1}^m b_j s^{\beta_j} + k + G(s)}. \quad (6.3)$$

Denote

$$L_j(s) = \frac{s^{\alpha_j-1}}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + \sum_{j=1}^m b_j s^{\beta_j} + k + G(s)}, \quad j = 0, 1, \dots, n, \\ M(s) = \frac{1}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + \sum_{j=1}^m b_j s^{\beta_j} + k + G(s)}, \quad (6.4)$$

then according to Theorem 3.2, their inverse Laplace transforms, namely  $l_j(t)$  and  $m(t)$ , belong to  $BSA_{2(\alpha_0-\alpha_j)+1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$ , and  $BSA_{2\alpha_0-1}(\mathbb{R}_+)$ , respectively. Moreover,

$$f(t) = f_0 l_0(t) + f_0 \sum_{j=1}^n a_j l_j(t) + \int_0^t m(t-\tau) h(\tau) d\tau. \quad (6.5)$$

Since  $m \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$ , and  $h \in L^1(\mathbb{R}_+)$ , by Lemma 3.1, their Laplace convolution  $m*h$  belongs to  $BSA_{2\alpha_0-1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$ . Hence,  $f$ , defined by (6.5), is from  $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$ . From (6.3) we have

$$F(s) \sim \frac{f_0}{s}, \quad s \rightarrow \infty.$$

Using the Tauberian theorem for the Laplace transform [23] we obtain

$$f(t) \sim f_0, \quad t \rightarrow 0+.$$

Consequently,  $f(0+) = f_0$ .

Conversely, let  $f$  be given by (6.5), where  $l_j, m$  are defined as the inverse Laplace transforms of (6.5). Then  $f \in BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$  and  $f(0+) =$

$f_0$ . Applying the Laplace transform to (6.5) and taking into account (6.4) we arrive at (6.3). Hence, (6.2) holds. The inverse Laplace transform of (6.2) yields (6.1).  $\square$

### Acknowledgements

The second author would like to thank the Vietnam Institute for Advanced Study in Mathematics VIASM for support during his visit to VIASM, and Dr. Huynh Van Ngai (Quy Nhon University), for fruitful discussions.

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*Received: November 20, 2020, Revised: April 28, 2021*

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Please cite to this paper as published in:

*Fract. Calc. Appl. Anal.*, Vol. **24**, No 3 (2021), pp. 739–754,  
DOI: 10.1515/fca-2021-0032